IV. *Researches on the Geometrical Properties of Elliptic Integrals.*

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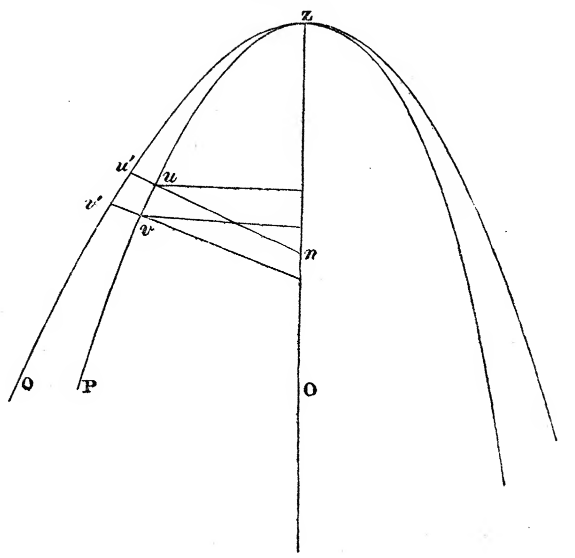
SECTION XI.—*On the Quadrature of the Logarithmic Ellipse and of the Logarithmic Hyperbola.*

LXXXIV. IN the former part of this paper, printed in the Philosophical Transactions for the year 1852, the author has shown that the geometrical types of those integrals, named by LEGENDRE and others elliptic functions, are the curves of symmetrical intersection of surfaces of the second order. In the progress of those investigations he discovered two curves, which he called the *Logarithmic Ellipse* and the *Logarithmic Hyperbola.* The properties of these curves have the same analogy to the paraboloid of revolution that spherical conics have to a sphere, or which ordinary conic sections bear to a plane. To determine the areas of those curves, or rather the portions of surface of the paraboloid bounded by them, appeared to the author a problem not undeserving of investigation.

Fig. 27.



The logarithmic ellipse is defined as the curve of intersection of a paraboloid of revolution with an elliptic cylinder whose axis coincides with that of the paraboloid.

The logarithmic hyperbola, in like manner, may be defined as the curve of intersection of a paraboloid of revolution with a cylinder whose base is an hyperbola, and whose axis coincides with that of the paraboloid.

Through the vertex Z of the paraboloid let two parabolas be drawn indefinitely near to each other, ZP, ZQ, and let two planes indefinitely near to each other at right angles to the axis OZ cut the parabolas in the points u, u', v, v'.

The little trapezoid $uvu'v'$ is the element of the surface, and if the normal un makes the angle μ with the axis OZ, $d\psi$ being the elementary angle between the planes, $uu'=k\tan\mu d\psi$, k being the semiparameter of the generating parabola.

Now $uv=ds=k\dfrac{d\mu}{\cos^3\mu}$. Hence the elementary trapezoid $uvu'v'=\dfrac{k^2\sin\mu d\mu d\psi}{\cos^4\mu}$.

Integrating this expression, area $=k^2\int d\psi\int\dfrac{\sin\mu}{\cos^4\mu}d\mu$; . . . . (436.)

or performing the integration with respect to μ,

$$\text{area}=\frac{k^2}{3}\int d\psi \sec^3\mu+\text{constant}.$$

Now when the area is 0, $\sec\mu=1$, and therefore

$$\text{constant}=-\frac{k^2}{3}\int d\psi. \quad \text{Whence}$$

$$\text{area}=\frac{k^2}{3}\int d\psi(\sec^3\mu-1). \qquad (437.)$$

This is the general expression for the surface of a paraboloid between two principal planes, and bounded by a curve.

When this curve is the logarithmic ellipse, let the area be put $(\Lambda\mathrm{H})$.

We must now express ψ and μ as functions of another variable θ.

Let $x=a\cos\theta$, $y=b\sin\theta$; the base of the cylinder being the ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. ψ is the angle which $\sqrt{x^2+y^2}$ makes with the axis a.

Now
$$\tan\psi=\frac{y}{x}=\frac{b}{a}\tan\theta, \qquad (438.)$$

and
$$d\psi=\frac{ab\,d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}. \qquad (439.)$$

But
$$\tan^2\mu=\frac{r^2}{k^2}=\frac{a^2\cos^2\theta+b^2\sin^2\theta}{k^2};$$

therefore
$$\sec^2\mu=\frac{(k^2+a^2)\cos^2\theta+(k^2+b^2)\sin^2\theta}{k^2}. \qquad (440.)$$

Hence substituting these values in (437.), we get for the area

$$(\Lambda\mathrm{H})=\frac{k^2}{3}\frac{ab}{k^3}\int\frac{d\theta\left[(k^2+a^2)\cos^2\theta+(k^2+b^2)\sin^2\theta\right]^{\frac{3}{2}}}{\left[a^2\cos^2\theta+b^2\sin^2\theta\right]}-\frac{k^2}{3}ab\int\frac{d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}. \qquad (441.)$$

Let
$$\frac{a^2-b^2}{a^2+k^2}=i^2,\quad \frac{a^2-b^2}{a^2}=e^2, \qquad (442.)$$

i being the modulus and e^2 the parameter, as in (15.).

The above expression may be written

$$\left.\begin{aligned}3d.(\Lambda\mathrm{H})=&\frac{ab}{k}\left[\frac{k^4d\theta}{(a^2\cos^2\theta+b^2\sin\theta)\sqrt{(k^2+a^2)-(a^2-b^2)\sin^2\theta}}\right]\\&+\frac{2ab}{k}\frac{k^2d\theta}{\sqrt{(k^2+a^2)-(a^2-b^2)\sin^2\theta}}\\&+\frac{ab}{k}\frac{(a^2\cos^2\theta+b^2\sin^2\theta)}{\sqrt{(k^2+a^2)-(a^2-b^2)\sin^2\theta}}\\&-k^2\frac{d\left(\frac{b}{a}\tan\theta\right)}{1+\left(\frac{b^2}{a}\tan\theta\right)^2}\end{aligned}\right\} \qquad (443.)$$

Therefore, integrating the preceding expression,

$$\left.\begin{aligned} 3(\Lambda\mathrm{H}) &= \frac{bk^3}{a\sqrt{k^2+a^2}}\int\frac{d\theta}{[1-e^2\sin^2\theta]\sqrt{1-i^2\sin^2\theta}} \\ &-\frac{2abk}{\sqrt{k^2+a^2}}\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}} \\ &+\frac{ab}{k}\sqrt{a^2+k^2}\int d\theta\sqrt{1-i^2\sin^2\theta}-k^2\tan^{-1}\left(\frac{b}{a}\tan\theta\right) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (444.)$$

Hence the area of the logarithmic ellipse, or rather the area of the paraboloid bounded by the logarithmic ellipse, may be expressed as a sum of elliptic integrals of the first, second and third orders, with a circular arc.

Since $\frac{a^2-b^2}{a^2} > \frac{a^2-b^2}{a^2+k^2}$, $e^2 > i^2$, or the function of the third order is of the circular form.

Assume a spherical conic section such that

$$\tan\alpha=\frac{a}{k},\quad \tan\beta=\frac{b}{k},\quad i^2=\frac{a^2-b^2}{a^2+k^2},$$

therefore $$\frac{\tan\beta}{\tan\alpha}\cos\alpha=\frac{bk}{a\sqrt{a^2+k^2}},\quad \sin^2\varepsilon=\frac{a^2-b^2}{a^2+k^2},\quad e^2=\frac{a^2-b^2}{a^2}.$$

Combining the first and last terms of the preceding equation, they become

$$-k^2\left[\tan^{-1}\left(\frac{b}{a}\tan\theta\right)-\frac{\tan\beta}{\tan\alpha}\cos\alpha\int\frac{d\theta}{[1-e^2\sin^2\theta]\sqrt{1-\sin^2\varepsilon\sin^2\theta}}\right].$$

Now this is the expression for the surface of a segment of a spherical ellipse whose principal angles are 2α and 2β*. Let this be S.

In the next place, $$k\sqrt{a^2+k^2}\int d\theta\sqrt{1-i^2\sin^2\theta}$$

is a portion of the elliptic cylinder whose altitude is k, and the semiaxes of whose base are $\sqrt{a^2+k^2}$ and $\sqrt{b^2+k^2}$. Let this be E,

and $$\frac{abk}{\sqrt{a^2+k^2}}\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}$$

is an expression for an arc of the spherical parabola whose focal distance is one-half the focal distance of the former. Let this be denoted by P.

Hence if we denote the entire surface round Z by $[\Lambda\mathrm{H}]$,

$$3[\Lambda\mathrm{H}]=4k\mathrm{E}+\frac{8abk}{\sqrt{b^2+k^2}}\mathrm{P}-4k^2\mathrm{S}. \quad \ldots\ldots \quad (445.)$$

Or the area of the logarithmic ellipse may be expressed as a sum of the arcs of a plane ellipse, of a spherical ellipse, and of a spherical parabola, multiplied by constant linear coefficients.

LXXXV. To find the area of the logarithmic hyperbola.

The general expression for the area, as in (437.), is $\frac{k^2}{3}\int(\sec^3\mu-1)d\psi$.

* See the Theory of Elliptic Integrals, and the Properties of Surfaces of the Second Order, applied to the investigation of the motion of a body round a fixed point, p. 16.

Now the equation of the base of the hyperbolic cylinder being $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

let $$x=a\sec\theta,\ y=b\tan\theta, \quad \ldots \quad (446.)$$

then $$\tan\psi=\frac{y}{x}=\frac{b}{a}\sin\theta,$$

and $$\frac{d\psi}{\cos^2\psi}=\frac{b}{a}\cos\theta d\theta,\ \cos^2\psi=\frac{a^2}{a^2+b^2\sin^2\theta};$$

hence $$d\psi=\frac{ab\cos\theta d\theta}{a^2+b^2\sin^2\theta}.$$

Since $$\tan^2\mu=\frac{r^2}{k^2}=\frac{a^2+b^2\sin^2\theta}{k^2\cos^2\theta},$$

$$\sec^2\mu=\frac{a^2+k^2\cos^2\theta+b^2\sin^2\theta}{k^2\cos^2\theta};$$

$$\because\quad \sec^3\mu=\frac{[k^2\cos^2\theta+a^2+b^2\sin^2\theta]^{\frac{3}{2}}}{k^3\cos^3\theta}.$$

Let $(\Lambda\Upsilon)$ denote the area of the logarithmic hyperbola, then

$$3(\Lambda\Upsilon)=k^2\int\frac{[k^2\cos^2\theta+a^2+b^2\sin^2\theta]^{\frac{3}{2}}ab\cos\theta d\theta}{k^3\cos^3\theta[a^2+b^2\sin^2\theta]}-k^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right) \quad \ldots \quad (447)$$

Let $$\mathrm{V}=k^2\cos^2\theta+a^2+b^2\sin^2\theta, \quad \ldots \quad (448.)$$

and the last equation will become

$$3(\Lambda\Upsilon)=\int\frac{abk^2\cos^2\theta d\theta}{[a^2+b^2\sin^2\theta]\sqrt{\mathrm{V}}}+\int\frac{2abkd\theta}{\sqrt{\mathrm{V}}}+\frac{ab}{k}\int\frac{[a^2+b^2\sin^2\theta]}{\cos^2\theta\sqrt{\mathrm{V}}}-k^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right);$$

and this may be written in the form

$$\left.\begin{aligned}3(\Lambda\Upsilon)=&\frac{ak^3(a^2+b^2)}{b}\int\frac{d\theta}{(a^2+b^2\sin^2\theta)\sqrt{\mathrm{V}}}+\left[2abk-\frac{ak^3}{b}-\frac{ab^3}{k}\right]\int\frac{d\theta}{\sqrt{\mathrm{V}}}\\&+\frac{ab}{k}(a^2+b^2)\int\frac{d\theta}{\cos^2\theta\sqrt{\mathrm{V}}}-k^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right)\end{aligned}\right\} \quad \ldots \quad (449.)$$

Let $$\frac{b^2}{a^2}=\tan^2\varepsilon=n,\quad \frac{k^2-b^2}{k^2+a^2}=i^2,$$

and the preceding equation may be written

$$\left.\begin{aligned}3(\Lambda\Upsilon)=&\frac{k^3(a^2+b^2)}{ab\sqrt{a^2+k^2}}\int\frac{d\theta}{[1+n\sin^2\theta]\sqrt{1-i^2\sin^2\theta}}\\&+\frac{ab(a^2+b^2)}{k\sqrt{a^2+k^2}}\int\frac{d\theta}{\cos^2\theta\sqrt{1-i^2\sin^2\theta}}-\frac{a}{bk}\ \frac{(k^2-b^2)^2}{\sqrt{a^2+k^2}}\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}-k^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right)\end{aligned}\right\}. \quad (450.)$$

Since $$n=\frac{b^2}{a^2},\quad \frac{1+n}{n}=\frac{a^2+b^2}{b^2},$$

and as $(1-m)(1+n)=1-i^2$, $m=\frac{k^2}{a^2+k^2}$, and (47.) gives

$$\left(\frac{1+n}{n}\right)\int\frac{d\theta}{\mathrm{N}\sqrt{\mathrm{I}}}-\left(\frac{1-m}{m}\right)\int\frac{d\theta}{\mathrm{M}\sqrt{\mathrm{I}}}=\frac{i^2}{mn}\int\frac{d\theta}{\sqrt{\mathrm{I}}}+\frac{1}{\sqrt{mn}}\tan^{-1}\left(\frac{\sqrt{mn}\sin\theta\cos\theta}{\sqrt{\mathrm{I}}}\right),$$

hence

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\theta}{\mathrm{N}\sqrt{\mathrm{I}}}=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\theta}{\mathrm{M}\sqrt{\mathrm{I}}}+\frac{i^2}{\sqrt{mn}}\int\frac{d\theta}{\sqrt{\mathrm{I}}}+\tan^{-1}\left[\frac{\sqrt{mn}\sin\theta\cos\theta}{\sqrt{\mathrm{I}}}\right]. \quad (451.)$$

But
$$\left(\frac{1+n}{n}\right)\sqrt{mn}=\frac{k(a^2+b^2)}{ab\sqrt{a^2+k^2}}.$$

Hence
$$\left.\begin{aligned}\frac{3(\Lambda\Upsilon)}{k^2}=&\frac{ab}{k\sqrt{k^2+a^2}}\int\frac{d\theta}{\left[1-\frac{k^2}{a^2+b^2}\sin^2\theta\right]\sqrt{\mathrm{I}}}\\&+\frac{ab(k^2-b^2)}{k^3\sqrt{k^2+a^2}}\int\frac{d\theta}{\sqrt{\mathrm{I}}}+\frac{ab(a^2+b^2)}{k^3\sqrt{k^2+a^2}}\int\frac{d\theta}{\cos^2\theta\sqrt{\mathrm{I}}}\\&+\tan^{-1}\left[\frac{\sqrt{mn}\sin\theta\cos\theta}{\sqrt{\mathrm{I}}}\right]-\tan^{-1}\left[\frac{b}{a}\sin\theta\right]\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (452.)$$

Now if Y be an arc of the plane hyperbola of which $\sqrt{k^2-b^2}$ is the transverse axis, and i the reciprocal of the eccentricity, we shall have

$$\frac{ab}{k^3}\mathrm{Y}=\frac{ab(a^2+b^2)}{k^3\sqrt{a^2+k^2}}\int\frac{d\theta}{\cos^2\theta\sqrt{\mathrm{I}}}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (453.)$$

And if we take the spherical ellipse whose principal semiangles, α and β, are given by the equations

$$\cos\alpha=\frac{b}{k},\ \cos\beta=\frac{b}{k}\sqrt{\frac{k^2+a^2}{k^2+b^2}},$$

we shall have
$$\sin^2\varepsilon=\frac{k^2-b^2}{k^2+a^2},\ e^2=\frac{k^2}{k^2+a^2}$$

and
$$\frac{\tan\beta}{\tan\alpha}\cos\alpha=\frac{ab}{k\sqrt{k^2+a^2}};$$

also
$$\psi=\tan^{-1}\left(\frac{b}{a}\sin\theta\right).$$

Hence the sum of the first and last terms may be written

$$\left[\psi-\frac{\tan\beta}{\tan\alpha}\cos\alpha\int\frac{d\theta}{[1-e^2\sin^2\theta]\sqrt{1-\sin^2\varepsilon\sin\theta}}\right];$$

and this expression is S, the value of the area of the spherical ellipse $(\alpha\beta)$, as shown at page 16 of the *Theory of Elliptic Integrals, &c.*

Now, as before, A being the transverse axe of the auxiliary hyperbola,

$$\mathrm{A}=\sqrt{k^2-b^2},\text{ and }\mathrm{B}=\sqrt{a^2+b^2},$$

hence the coefficient of $\int\frac{d\theta}{\sqrt{\mathrm{I}}}$ may be written $\frac{ab}{k^3}\frac{\mathrm{A}^2}{\mathrm{B}}j$, and the equation (452.) finally assumes the form

$$3k(\Lambda\Upsilon)=ab\left[\mathrm{Y}+\frac{\mathrm{A}^2}{\mathrm{B}}j\int\frac{d\theta}{\sqrt{\mathrm{I}}}\right]-k^2\mathrm{S}+k^3\tan^{-1}\left[\frac{\sqrt{mn}\sin\theta\cos\theta}{\sqrt{\mathrm{I}}}\right]. \quad . \; . \; . \quad (454.)$$

Or the area of the logarithmic hyperbola may be expressed as a sum of the arcs of

a common hyperbola, of a spherical ellipse, of a spherical parabola and of a circular arc, multiplied by constant coefficients.

LXXXVI. There is one particular case when the area of the logarithmic hyperbola may be represented by a very simple expression. Let $k=b$, then if we turn to (448.) $V=a^2+b^2$, and $I=1$, since $i=0$. Hence (452.) may be changed into

$$3(\Lambda\Upsilon)=a\sqrt{a^2+b^2}\tan\theta+b^2\tan^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\tan\theta\right)$$

$$+b^2\tan^{-1}\left(\frac{b^2}{a\sqrt{a^2+b^2}}\sin\theta\cos\theta\right)-b^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right);$$

and this expression may be reduced to

$$3(\Lambda\Upsilon)=a\sqrt{a^2+b^2}\tan\theta+b^2\tan^{-1}\left(\frac{\sqrt{a^2+b^2}}{a}\tan\theta\right)-b^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right), \quad \text{(455.)}$$

a value entirely independent of elliptic integrals, and which may be represented by a right line and the difference of two circular arcs.

LXXXVII. The curve of symmetrical intersection of a sphere by a paraboloid, whose principal sections are unequal, may be rectified by an elliptic integral of the third order and circular form.

Let
$$x^2+y^2+z^2=2rz, \text{ and } \frac{x^2}{k}+\frac{y^2}{k_1}=2z \quad \text{(456.)}$$

be the equations of the sphere and paraboloid. Then finding the values of dx, dy and dz,

$$\left(\frac{ds}{dz}\right)^2=\frac{(r^2-kk_1)z-2r(r-k)(r-k_1)}{z[z-2(r-k)][2(r-k_1)-z]}. \quad \text{(457.)}$$

Assume
$$z=2(r-k)\cos^2\theta+2(r-k_1)\sin^2\theta. \quad \text{(458.)}$$

Introducing the new variable θ and its functions,

$$\frac{ds}{d\theta}=2\frac{\sqrt{k_1(r-k)^2+k(r-k_1)^2\tan^2\theta}}{\sqrt{(r-k)+(r-k_1)\tan^2\theta}}. \quad \text{(459.)}$$

Assume
$$k(r-k_1)^2\tan^2\theta=k_1(r-k)^2\tan^2\phi, \quad \text{(460.)}$$

then introducing the variable ϕ and its functions,

$$\frac{ds}{d\theta}=\frac{\sqrt{kk_1}\sqrt{(r-k)(r-k_1)}}{\sqrt{k(r-k_1)\cos^2\phi+k_1(r-k)\sin^2\phi}}$$

$$=\frac{\sqrt{k_1(r-k)}}{\sqrt{1-\frac{r}{k}\left(\frac{k-k_1}{r-k_1}\right)\sin^2\phi}},$$

and
$$\frac{d\theta}{d\phi}=\frac{\sqrt{k_1}(r-k)}{\sqrt{k}(r-k_1)\left[1-\frac{(k-k_1)}{k}\frac{(r^2-kk_1)}{(r-k_1)^2}\sin^2\phi\right]}. \quad \text{(461.)}$$

Hence integrating,
$$s=\frac{k_1(r-k)^{\frac{3}{2}}}{\sqrt{k}(r-k_1)}\int\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad \text{(462.)}$$

If we write m for $\frac{k-k_1}{k}\left(\frac{r^2-kk_1}{(r-k_1)^2}\right)$, and i^2 for $\frac{k-k_1}{k}\frac{r}{r-k_1}$.

Now as $$i^2=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}, \text{ and } m=e^2=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha\cos^2\beta}, \quad . \quad . \quad . \quad . \quad (463.)$$

we get from these equations

$$\tan^2\alpha=\frac{k(r-k_1)}{r(r-k)}, \ \tan^2\beta=\frac{k_1(r-k)}{r(r-k_1)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (464.)$$

whence $$\sqrt{r^2-kk_1}\,\frac{\tan\beta}{\tan\alpha}\sin\beta=\frac{k_1(r-k)^{\frac{3}{2}}}{\sqrt{k(r-k_1)}}.$$

Making these substitutions, (462.) will become

$$s=\sqrt{r^2-kk_1}\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\phi}{[1-e^2\sin^2\phi]\sqrt{1-\sin\eta\sin\phi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (465.)$$

Now, as we have shown in (16.), this expression denotes an arc of the spherical ellipse whose principal angles are given by the equations (464.), and whose radius is $\sqrt{r^2-kk_1}$. Hence if a sphere be described whose radius is not r, but $\sqrt{r^2-kk_1}$, the length of the curve, the intersection of the sphere (r) with the paraboloid (kk_1) will be equivalent to that of a spherical ellipse described on the sphere whose radius is $\sqrt{r^2-kk_1}$.

When $r=k$, k being greater than k_1, (459.) becomes

$$\frac{ds}{d\theta}=2\sqrt{k(k-k_1)} \text{ or } s=2k(k-k_1)\theta.$$

Hence s is an arc of a circle. That such ought to be the case is manifest, for in this case the sphere intersects the paraboloid in its circular sections, and $\sqrt{\frac{k-k_1}{k}}$ is the cosine of the angle which the plane of the circular section of the paraboloid makes with its axis.

We have shown in the first part of this paper that the curves of intersection of *concentric* surfaces of the second order may be rectified by elliptic integrals. When the intersecting surfaces are not concentric, the rectification of the curve of intersection may be reduced to the integration of an expression which may be called an hyperelliptic integral.

The general expression for the length of an arc of this curve will be an integral of the form

$$s=\int dx\sqrt{\frac{\alpha x^4+\beta x^3+\gamma x^2+\delta x+\varepsilon}{ax^4+bx^3+cx^2+ex+f}}.$$

When the surfaces are symmetrically placed and have a common plane of contact, the above expression may be reduced to

$$s=\int dx\sqrt{\frac{\alpha x^3+\beta x^2+\gamma x+\delta}{ax^3+bx^2+cx+e}}.$$

This is also an hyperelliptic integral.

When, moreover, the surfaces are concentric and symmetrically placed, the preceding expression may still further be simplified to

$$s=\int dx\sqrt{\frac{\alpha x^2+\beta x+\gamma}{ax^2+bx+c}},$$

which is the general form for elliptic integrals.

We can perceive therefore that the solution of the general problem, to determine the length of the curve in which two surfaces of the second order may intersect, investigated under its most general form, far transcends the present powers of analysis. It is only when one of the surfaces becomes a plane, or when they are concentric and symmetrically placed, that the problem under these restricted conditions admits of a complete solution.

We may hence also surmise how vast are the discoveries which still remain to be explored in the wide regions of the integral calculus. We see how questions which arise from the investigation of problems based on the most elementary geometrical forms—surfaces of the second order—baffle the utmost powers of a refined analysis, with all the aids of modern improvement. It is not a little curious, that nearly all the branches of modern analysis, such as plane and spherical trigonometry, the doctrine of logarithms and exponentials, with the theory of elliptic integrals, may all be derived from the investigation of one geometrical problem,—To determine the length of an arc of the intersecting curve of two surfaces of the second order.

LXXXVIII. In the logarithmic hyperconic sections, we may develope properties analogous to those found in the spherical and plane sections, if we substitute parabolic arcs for arcs of great circles in the one, and for right lines in the other. Here follow a few of those theorems.

1. From any point on a parabolic section of the paraboloid let two parabolas be drawn touching the logarithmic ellipse or the logarithmic hyperbola, the parabolic arcs joining the points of contact will all pass through one point on the surface of the paraboloid.

2. If a hexagon, whose sides are parabolic arcs, be inscribed in a logarithmic ellipse or logarithmic hyperbola, the opposite parabolic arcs will meet two by two on a parabola.

3. If a hexagon, whose sides are parabolas, be circumscribed to a logarithmic ellipse, the parabolic arcs joining the opposite vertices will pass through a fixed point on the surface of the paraboloid.

4. If through the centre of a logarithmic ellipse or logarithmic hyperbola two parabolic arcs are drawn at right angles to each other, meeting the curve in two points, and parabolic arcs be drawn touching the curve in these points, they will meet on another logarithmic ellipse or logarithmic hyperbola.

5. If a circle, whose radius is a, be described on the surface of the paraboloid, and therefore touching the logarithmic ellipse or the logarithmic hyperbola at the extremities of its major axis, and from the extremities of any diameter two parabolic arcs

be drawn to any third point on the circle, if one of these parabolic arcs touches the logarithmic ellipse or the logarithmic hyperbola, the other will pass through a fixed point on the surface of the paraboloid.

6. If on the paraboloid we describe a circle whose radius is $\sqrt{a^2 \pm b^2}$, and if from the extremities of any diameter of this circle we draw parabolic arcs touching the logarithmic ellipse or the logarithmic hyperbola, these tangent parabolic arcs will meet on the circle.

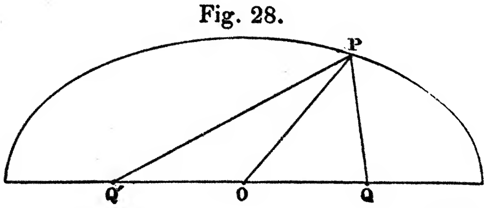
These theorems will suffice. There would be little difficulty in extending the list. In fact nearly all the projective properties of right lines and conic sections on a plane may be transformed into analogous properties of great circles and spherical conic sections on the surface of a sphere, and of parabolic arcs and logarithmic sections on the surface of a paraboloid.

SECTION XII.—*On the Rectification of the Lemniscates.*

LXXXIX. There is a particular class of plane curves, of which the lemniscate of BERNOULLI is an example, to which the principles established in the foregoing pages may be applied with much elegance.

Definition.—This entire class of curves may be defined by the following property. The square of the rectangle under the radii vectores drawn from the foci to any point on the curve is equal to a constant, plus or minus the square of the semidiameter multiplied by a constant quantity.

Fig. 28.



Let Q, Q' be the foci, and O the centre, ρ, $\rho_{\prime}$ r the lines drawn from these points to any point on the curve. Let $OQ=OQ'=c$, and let f be a variable constant.

Then by the definition

$$\rho^2\rho_{\prime}^2=c^4\pm f^2r^2. \qquad (466.)$$

But
$$\rho^2\rho_{\prime}^2=(x^2+y^2)^2+c^4+2c^2y^2-2c^2x^2,$$

and
$$r^2=x^2+y^2,$$

hence
$$(x^2+y^2)^2=(f^2+2c^2)x^2+(f^2-2c^2)y^2. \qquad (467.)$$

This is the general equation of the curve, which assumes different forms, as we assign varying values to f and c. Some examples may be given.

(a.) Let $c=0$, or $f=\infty$, the equation is that of a circle.

(b.) Let $f^2>2c^2$, and make $f^2+2c^2=a^2$, $f^2-2c^2=b^2$, $\qquad$ (468.)

the equation will become
$$(x^2+y^2)^2=a^2x^2+b^2y^2. \qquad (469.)$$

This is the equation of a curve which may be called the elliptic lemniscate. It is the locus, as is well known, of the intersection of central perpendiculars with tangents to an ellipse.

(c.) Let $f^2=2c^2$. The equation becomes $(x^2+y^2)^2=4c^2x^2$, or the equation is that of two equal circles in external contact.

(d.) Let $f^2<2c^2$. The equation becomes

$$(x^2+y^2)^2=(2c^2+f^2)x^2-(2c^2-f^2)y^2\text{; and } a^2>b^2.$$

(e.) Let $f^2=0$. The equation becomes $(x^2+y^2)^2=2c^2(x^2-y^2)$, or the equation is that of the lemniscate of BERNOULLI.

(f.) Let f^2, passing through 0, be taken with a negative sign. The equation in this case becomes

$$(x^2+y^2)^2=(2c^2-f^2)x^2-(2c^2+f^2)y^2\text{, and } b^2>a^2.$$

In one case only does the equation of the lemniscate in its general form coincide with that of CASSINI's ellipse; namely, when $f=0$, and $h=c$, h^2 being the product of the radii vectores from the foci.

The definition of CASSINI's ellipse being "a curve such that the product of the radii vectores drawn from two fixed points—the foci—to a third point on the curve, shall be constant and equal to h^2," its equation will obviously be, $2c$ being the distance between the foci,

$$h^4-c^4=(x^2+y^2)^2-2c^2(x^2-y^2),$$

when $$h=c,\quad (x^2+y^2)^2=2c^2(x^2-y^2).$$

This is the equation of the lemniscate of BERNOULLI.

These elliptic lemniscates may also be defined as the orthogonal projections of the curves of symmetrical intersection of a paraboloid of revolution with cones of the second degree, having their centres at the vertex of the paraboloid. Let α and β be the principal semiangles of one of the cones. Its equation is

$$\cot^2\alpha . x^2 \pm \cot^2\beta . y^2=z^2.$$

Make $\tan\alpha=\frac{2k}{a}$, $\tan\beta=\frac{2k}{b}$, and the equation of the cone becomes

$$a^2x^2\pm b^2y^2=4k^2z^2.$$

Let the equation of the paraboloid be $x^2+y^2=2kz$.

Eliminating z, the equation of the projection of the curve of intersection will become

$$(x^2+y^2)^2=a^2x^2\pm b^2y^2.$$

XC. When the section is an ellipse, the equation of this curve is, as in (469.),

$$(x^2+y^2)^2=a^2x^2+b^2y^2. \qquad (470.)$$

This equation may be put in the form $r^2=a^2\cos^2\lambda+b^2\sin^2\lambda$,

r being the radius vector, and λ the angle it makes with the axis. Let s be an arc of this curve;

since $$\left(\frac{ds}{d\lambda}\right)^2=r^2+\left(\frac{dr}{d\lambda}\right)^2,\ \left(\frac{ds}{d\lambda}\right)^2=\frac{a^4\cos^2\lambda+b^4\sin^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda}.$$

Assume $$\tan\lambda = \frac{a}{b}\tan\phi. \quad \text{(471.)}$$

Making in the last equation the substitutions suggested by this transformation, we get, after some reductions,

$$b\frac{ds}{d\phi} = \frac{a^2\cos^2\phi + b^2\sin^2\phi}{\left[1+\left(\frac{a^2-b^2}{b^2}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\phi}}. \quad \text{(472.)}$$

Let $$\frac{a^2-b^2}{b^2} = n, \quad \frac{a^2-b^2}{a^2} = i^2, \quad \text{(473.)}$$

and the last equation will become

$$s = \frac{a^2+b^2}{b}\int\frac{d\phi}{[1+n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}} - b\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}. \quad \text{(474.)}$$

On the plane ellipse as a base, let a vertical cylinder be erected, and from the centre of this ellipse let a sphere with a radius $=\sqrt{a^2+b^2}$ be described. This sphere will cut the elliptic cylinder in a spherical conic section. The expression for an arc of a spherical conic section measured from the extremity of the *minor arc* is given by the equation

$$\frac{\bar{s}}{\mathrm{R}} = \frac{\cos\beta}{\cos\alpha\sin\alpha}\int\frac{d\phi}{[1+\tan^2\varepsilon\sin^2\phi]\sqrt{1-\sin^2\eta\sin^2\phi}} - \frac{\cos\alpha\cos\beta}{\sin\alpha}\int\frac{d\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}.$$

See Theory of Elliptic Integrals, p. 27.

Now in this case α and β being the principal angles of the concentric cone whose base is the spherical conic section,

$$\sin^2\alpha = \frac{a^2}{a^2+b^2}, \quad \sin^2\beta = \frac{b^2}{a^2+b^2},$$

therefore $$\cos^2\alpha = \frac{b^2}{a^2+b^2}, \quad \cos^2\beta = \frac{a^2}{a^2+b^2}.$$

Hence $$\alpha+\beta = \frac{\pi}{2},$$

or the sum of the principal angles of this cone is equal to two right angles, or the cone is its own supplemental cone. From these equations we may infer that

$$\frac{\cos\beta}{\cos\alpha\sin\alpha} = \frac{\sqrt{a^2+b^2}}{b}, \quad \frac{\cos\alpha\cos\beta}{\sin\alpha} = \frac{b}{\sqrt{a^2+b^2}}$$

$$\tan^2\varepsilon = \frac{\sin^2\alpha - \sin^2\beta}{\cos^2\alpha} = \frac{a^2-b^2}{b^2}, \quad \sin^2\eta = \frac{\sin\alpha - \sin^2\beta}{\sin\alpha} = \frac{a^2-b^2}{a^2}.$$

Making these substitutions in the preceding equation, we get

$$\bar{s} = \frac{a^2+b^2}{b}\int\frac{d\phi}{\left[1+\left(\frac{a^2-b^2}{b^2}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\phi}} - b\int\frac{d\phi}{\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\phi}}.$$

On comparing this equation with (474.) we shall see that they are precisely identical. Whence we infer that an arc of the elliptic lemniscate is equal to an arc of a spherical ellipse which is self-supplemental. It is very remarkable, that, whatever be the ratio of a to b the semiaxes of the plane ellipse or of the elliptic lemniscate, the arc is always equal to an arc of this particular species of spherical ellipse.

There is another property of this spherical ellipse, that its area, together with twice the lateral surface of the cone, is equal to a hemisphere. See Theory of Elliptic Integrals, &c., p. 21.

XCI. We may obtain under another form an expression for the arc of an elliptic lemniscate.

Let the polar angle λ be measured from the minor axis of the curve. Its equation in this case will be

$$r^2=a^2\sin^2\lambda+b^2\cos^2\lambda,$$

$$\therefore \quad \frac{ds^2}{d\lambda^2}=\frac{a^4\sin^2\lambda+b^4\cos^2\lambda}{a^2\sin^2\lambda+b^2\cos^2\lambda}.$$

Assume

$$\tan\lambda=\frac{b^2}{a^2}\tan\psi, \quad \ldots \quad (475.)$$

hence

$$\frac{ds^2}{d\lambda^2}=\frac{a^2b^2}{a^2\cos^2\psi+b^2\sin^2\psi}, \text{ and } \frac{d\lambda}{d\psi}=\frac{a^2b^2}{a^4\cos^2\psi+b^4\sin^2\psi},$$

integrating,

$$s=\frac{b^3}{a^2}\int\frac{d\psi}{\left[1-\left(\frac{a^4-b^4}{a^4}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\psi}}. \quad \ldots \quad (476.)$$

Let, as before, a cylinder be erected on the ellipse and the sphere described from its centre with a radius equal to $\sqrt{a^2+b^2}$, it will cut the cylinder in a spherical ellipse, whose arc is given by the integral

$$\frac{\bar{s}}{k}=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\psi}{\left[1-\left(\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{\sin^2\alpha-\sin\beta}{\sin^2\alpha}\right)\sin^2\psi}}.$$

Now since

$$\sin^2\alpha=\frac{a^2}{a^2+b^2}, \quad \sin^2\beta=\frac{b^2}{a^2+b^2}$$

$$k\frac{\tan\beta}{\tan\alpha}\sin\beta=\frac{b^3}{a^2}, \quad \frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=\frac{a^4-b^4}{a^4}, \quad \frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=\frac{a^2-b^2}{a^2},$$

substituting, we obtain

$$\bar{s}=\frac{b^3}{a^2}\int\frac{d\psi}{\left[1-\left(\frac{a^4-b^4}{a^4}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\psi}}.$$

Now this is precisely the same equation as (476.), whence we infer that the arc of the elliptic lemniscate is equal to an arc of a self-supplemental spherical ellipse. Writing m for the parameter in this expression, we can easily show that the parameters in this and the preceding formula (474.) are *conjugate* parameters. The con-

dition of the conjugation of parameters in the circular form is

$$(1+n)(1-m)=(1-i^2).$$

Now $\qquad 1+n=\frac{a^2}{b^2},\ 1-m=\frac{b^4}{a^4},\ 1-i^2=\frac{b^2}{a^2},$

whence the proposition is manifest.

An invariable relation exists between the parameter m and the modulus i; for as

$$m=\frac{a^4-b^4}{a^4} \text{ and } i^2=\frac{a^2-b^2}{a^2} \quad \frac{m}{i^2}+i^2=2, \quad \ldots \ldots \quad (477.)$$

hence i being given, m also is given; or the elliptic lemniscate can be represented by only one species of spherical ellipse, that in which the sum of the principal arcs is equal to two right angles.

On the Hyperbolic Lemniscate.

XCII. The equation of the lemniscate in this case is

$$(x^2+y^2)^2=a^2x^2-b^2y^2.$$

Following the steps indicated in (XC.), we find

$$\frac{ds^2}{d\lambda^2}=\frac{a^4\cos^2\lambda+b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda};$$

the limits of λ are 0 and $\tan^{-1}\frac{a}{b}$.

Assume $\qquad \sin^2\lambda=\frac{a^4\sin^2\phi}{a^2b^2+a^4\sin^2\phi+b^4\cos^2\phi}. \quad \ldots \ldots \quad (478.)$

The limits of ϕ, corresponding to $\lambda=0$ and $\lambda=\tan^{-1}\frac{a}{b}$, are $\phi=0$, and $\phi=\frac{\pi}{2}$.

Substituting this value of $\sin^2\lambda$ in the preceding equation, we shall find

$$\frac{ds}{d\lambda}=\frac{a}{\cos\phi}. \quad \ldots \ldots \quad (479.)$$

From (478.) we may derive

$$\frac{d\lambda}{d\phi}=\frac{a^2b(a^2+b^2)\cos\phi}{[a^2b^2+a^4\sin^2\phi+b^4\cos^2\phi]\sqrt{a^2+b^2\cos^2\phi}}.$$

Multiplying the two latter equations together and reducing, we get

$$s=\frac{a^3}{b\sqrt{a^2+b^2}}\int\frac{d\phi}{\left[1+\left(\frac{a^2-b^2}{b^2}\right)\sin^2\phi\right]\sqrt{1-\frac{b^2}{a^2+b^2}\sin^2\phi}}. \quad \ldots \ldots \quad (480.)$$

When $a=b$, or when the lemniscate is that of BERNOULLI, we get the well-known expression

$$s=\frac{a}{\sqrt{2}}\int\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}.$$

When $a>b$ the integral is of the third order and *circular* form, but when $a<b$ the integral is of the third order and *logarithmic* form. That it is of the logarithmic form may thus be shown.

Let $$\frac{b^2-a^2}{b^2}=m, \text{ and } i^2=\frac{b^2}{a^2+b^2}.$$

Hence $$i^2-m=\frac{a^4}{b^2(a^2+b^2)};$$

or i^2 is greater than m; but we know that the form is logarithmic when the square of the modulus is greater than the parameter, when it is affected with a negative sign.

This is a result truly remarkable. All analysts know the impossibility of transforming the circular form into the logarithmic, or *vice versâ*, by any other than an imaginary transformation. The utmost efforts of the most accomplished analysts have been exhausted in the attempt; yet in this particular case their geometrical connection is very close. The modulus and the parameter are connected by the equation

$$\frac{1}{i^2}\mp m=2; \qquad \text{(481.)}$$

the upper sign to be taken in the circular form, the lower in the logarithmic.

There are two distinct cases to be considered; when a is greater than b, and when a is less than b.

Case I. $a>b$.

Let a plane ellipse be constructed whose principal semiaxes A and B are given by the equations

$$A^2=a^2+b^2, \quad B^2=a^2, \qquad \text{(482.)}$$

and let a sphere be described from the centre of this ellipse with a radius

$$=\frac{a^2}{\sqrt{a^2-b^2}}=\frac{B^2}{\sqrt{2B^2-A^2}}=R.$$

Then we can find, as follows, the length of an arc of the spherical ellipse, the intersection of the sphere whose radius is R, with the cylinder standing on the ellipse whose semiaxes are A and B.

Since $$\sin^2\alpha=\frac{A^2}{R^2}=\frac{a^4-b^4}{a^4}, \quad \cos^2\alpha=\frac{b^4}{a^4},$$

and $$\sin^2\beta=\frac{B^2}{R^2}=\frac{a^2-b^2}{a^2}, \quad \cos^2\beta=\frac{b^2}{a^2},$$

$$\therefore \quad \frac{R\cos\beta}{\cos\alpha\sin\alpha}=\frac{a^5}{b(a^2-b^2)\sqrt{a^2+b^2}}.$$

We have also

$$\left.\begin{aligned} \frac{R\cos\beta\cos\alpha}{\sin\alpha}&=\frac{ab^3}{(a^2-b^2)\sqrt{a^2+b^2}} \\ \tan^2\varepsilon=\frac{\cos^2\beta-\cos^2\alpha}{\cos^2\alpha}&=\frac{a^2-b^2}{b^2} \\ i^2=\sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}&=\frac{b^2}{a^2+b^2} \end{aligned}\right\} \qquad \text{(483.)}$$

Substituting these values in (46.) the expression for an arc of a spherical ellipse with a *positive* parameter, and writing $\bar{s}$ for the arc, we get

$$\left.\begin{aligned}\frac{a^2-b^2}{a^2}\bar{s}=&\frac{a^3}{b\sqrt{a^2+b^2}}\int\frac{d\phi}{\left[1+\left(\frac{a^2-b^2}{b^2}\right)\sin^2\phi\right]\sqrt{1-\frac{b^2}{a^2+b^2}\sin^2\phi}}\\&-\frac{b^3}{a\sqrt{a^2+b^2}}\int\frac{d\phi}{\sqrt{1-\frac{b^2}{a^2+b^2}\sin^2\phi}}-\sqrt{a^2-b^2}\tau.\end{aligned}\right\} \quad (484.)$$

Comparing this with (480.), we find

$$s-\left(\frac{a^2-b^2}{a^2}\right)\bar{s}=\frac{b^3}{a\sqrt{a^2+b^2}}\int\frac{d\phi}{\sqrt{1-\left(\frac{b^2}{a^2+b^2}\right)\sin^2\phi}}+\sqrt{a^2-b^2}\tau,$$

or the *difference* between an arc of a hyperbolic lemniscate and an arc of a spherical ellipse may be expressed by an integral of the first order, together with a circular arc. When $a=b$, the radius of the sphere is infinite, the sphere becomes a plane, so that it is not possible to express an arc of a spherical ellipse by the common lemniscate.

Case II. Let $b>a$.

In this case the arc of the hyperbolic lemniscate may be expressed by an arc of a logarithmic ellipse of a *particular species*, or one whose parameter and modulus are connected by the relation given in (481.).

Resuming the expression in (480.) for the arc of the hyperbolic lemniscate,

$$s=\frac{a^3}{b\sqrt{a^2+b^2}}\int\frac{d\phi}{\left[1-\left(\frac{b^2-a^2}{b^2}\right)\sin^2\phi\right]\sqrt{1-\frac{b^2}{b^2+a^2}\sin^2\phi}}.$$

Let
$$\left.\begin{aligned}&\frac{b^2-a^2}{b^2}=m, \quad \frac{b^2}{b^2+a^2}=i^2,\\ &m+n-mn=i^2, \quad n=\frac{a^2}{a^2+b^2}\end{aligned}\right\} \quad (485.)$$
then as

Let A and B be the semiaxes of the base of the elliptic cylinder, k the parameter of the paraboloid whose intersection with the cylinder gives the logarithmic ellipse. Assume for the principal semimajor axis of the elliptic base

$$\mathrm{A}=\sqrt{a^2+b^2}. \quad (486.)$$

In (171.) we found the following relations between A, B, k, m, n,

$$\frac{\mathrm{A}^2}{k^2}=\frac{mn(1-n)}{(n-m)^2}, \quad \frac{\mathrm{B}^2}{k^2}=\frac{mn(1-m)}{(n-m)^2},$$

and as we assume $\mathrm{A}=\sqrt{a^2+b^2}$, we get, substituting for m and n their values in terms of a and b, the semiaxes of the hyperbola

$$\mathrm{B}=\frac{b^2}{a}, \text{ and } k=\frac{a^2b^2+a^4-b^4}{a^2\sqrt{b^2-a^2}}. \quad (487.)$$

In (163.) we found for the equation of the logarithmic ellipse measured from the *minor* axis, and multiplied by the indeterminate factor Q,

$$\left.\begin{aligned}2Q\Sigma=&-\left(\frac{1-m}{m}\right)\sqrt{mnk}Q\int\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\\&+\frac{kQ\sqrt{mn}}{n-m}\left[\int d\phi\sqrt{\mathrm{I}}+\left(\frac{i^2-m}{m}\right)\int\frac{d\phi}{\sqrt{\mathrm{I}}}-m\Phi\right]\end{aligned}\right\} \quad (488.)$$

If in this equation we substitute for m, n, and k their values as given in (485.), and equate the coefficient $\left(\frac{1-m}{m}\right)\sqrt{mnk}Q$ with the coefficient $\frac{a^3}{b\sqrt{a^2+b^2}}$ of the expression for the lemniscate in (480.), we shall find

$$Q=\frac{a^2(b^2-a^2)}{a^2b^2+a^4-b^4};$$

hence the last equation, substituting this value of Q, will become

$$\begin{aligned}\frac{2a^2(b^2-a^2)}{a^2b^2+a^4-b^4}\Sigma+s=&\frac{ab(b^2-a^2)\sqrt{a^2+b^2}}{a^2b^2+a^4-b^4}\int d\phi\sqrt{\mathrm{I}}\\&+\frac{a^5b}{[a^2b^2+a^4-b^4]\sqrt{a^2+b^2}}\int\frac{d\phi}{\sqrt{\mathrm{I}}}-\frac{a(b^2-a^2)^2\sqrt{a^2+b^2}}{b(a^2b^2+a^4-b^4)}\Phi; \quad (489.)\end{aligned}$$

or the *sum* of an arc of a hyperbolic lemniscate and of an arc of a logarithmic ellipse may be expressed as a sum of integrals of the first and second orders with a circular arc.

When $b=a$, the above expression will become

$$s=\frac{a}{\sqrt{2}}\int\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}.$$

In this case the parameter of the paraboloid becomes infinite, and therefore the paraboloid a plane, just as the sphere became a plane in the last case; so that we cannot express integrals of the third order, whether circular or logarithmic, by an arc of a common lemniscate.

XCIII. FAGNANI, the Italian geometer, first showed that the lemniscate of the equilateral hyperbola might be rectified by an elliptic integral of the first order whose modulus is $\frac{1}{\sqrt{2}}$. He did not however extend his researches to the investigation of the general problem of the rectification of the lemniscates.

Although the lemniscates may be rectified by elliptic integrals of the third order, as well circular as logarithmic, yet these curves cannot be accepted as general representatives of integrals of the third order, because in the functions which represent those curves, the parameters and the moduli are connected by an invariable relation, as in (477.) and (481.). Thus the elliptic lemniscate, whatever be the ratio of the axes of the generating plane ellipse, can be represented only by a particular species of spherical ellipse, that whose principal arcs are supplemental.

XCIV. The general fundamental expressions for the rectification of curve lines, whether of single or double flexion, show that the arc of a curve may in general be represented as the sum of two quantities, an integrated and a non-integrated part, or as the proposition may be more briefly put, an arc of a curve may be expressed as the sum of an integral and a residual. Thus the arc of a plane ellipse is equal to an integral and a residual, which latter is a right line. An arc of a parabola is the sum of an integral and a residual, which latter is also a right line. An arc of a spherical ellipse is the sum of an integral and a residual, the latter being an arc of a circle, while an arc of a logarithmic ellipse is made up of two portions, one a sum of integrals, the other—the residual—being an arc of a common parabola. It appears therefore to be an expenditure of skill in a wrong direction to devise curves whose arcs should differ from the corresponding arcs of hyperconic sections by the above-named residuals. Thus geometers have sought to discover plane curves whose arcs should be represented by elliptic integrals of the first order, without any residual quantity—the common lemniscate for example, when the modulus has a particular value. It is possible that such may be found. In the same way, an exponential* curve may be devised, whose arc shall be represented by the integral $k\int\frac{d\theta}{\cos\theta}$, instead of taking it with the residual quantity $k\tan\theta\sec\theta$, as the expression for an arc of a common parabola. Thus geometers have been led to look for the types of elliptic integrals among the higher orders of plane curves, overlooking the analogy which points to the intersection of surfaces of the second order as the natural geometrical types of those integrals.

* The equation of this exponential curve is $e^{\frac{y}{k}}\cos\left(\frac{x}{k}\right)=1$. It is easily seen that when $x=0$, $y=0$, also. And when $x=\frac{k\pi}{2}$, $y=\infty$. Hence the curve passes through the origin and has asymptots parallel to the axis of y at the distance $\frac{k\pi}{2}$ from the origin.

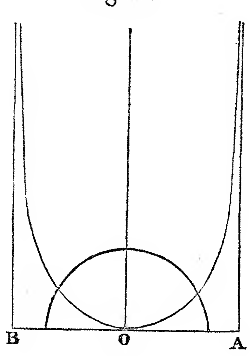
If we substitute for $\cos\left(\frac{x}{k}\right)$ its exponential expression $\frac{e^{\frac{x}{k}\sqrt{-1}}+e^{-\frac{x}{k}\sqrt{-1}}}{2}$, the equation of the curve becomes $\left[e^{\frac{y+x\sqrt{-1}}{k}}-1\right]+\left[e^{\frac{y-x\sqrt{-1}}{k}}-1\right]=0.$

The common equation of the circle $y^2+x^2=k^2$, may be written

$$\log\left[\frac{y+x\sqrt{-1}}{k}\right]+\log\left[\frac{y-x\sqrt{-1}}{k}\right]=0.$$

In this form the similarity of the equations of the exponential curve and the circle is evident.

Fig. 29.



In the equation $s=k\int\frac{d\theta}{\cos\theta}$, if we make the imaginary transformation $\tan\theta=\sqrt{-1}\sin\omega$, the resulting expression will be $s=k\omega\sqrt{-1}$, or the expression is transformed from a logarithmic to a circular function.